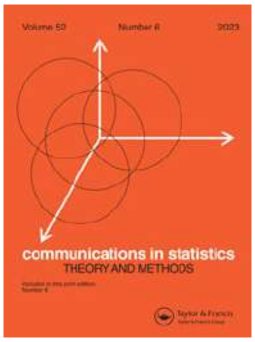

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Nonexistence of robust designs against presence of more than one outlier in a restricted class

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ABSTRACT

Presence of one or more outliers in the observations affect inference procedure in statistical analysis. Designs robust against presence of a single outlier can be found in the literature for both regression and block design set-ups. In this paper, an attempt has been made to find a robust design in a block design set-up for the estimation of a full set of orthonormal treatment contrasts when there are more than one outlier among the observations. It appears that symmetry and balance play an important role in this study; it is seen that as we deviate from them, designs deviate from robustness.

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1. Introduction

An outlier in a data set is an observation that differs significantly from other observations. An outlier may be due to variability in the measurement or it may indicate experimental error; the latter are sometimes excluded from the data set. An outlier can cause serious problems in statistical analyses. Box and Draper (1975) first considered the problem of finding designs robust against the presence of one or more outliers in the observations for the estimation of parameters of a regression experiment. Mandal (1989) extended it to the block design (BD) set-up for the estimation of a full set of orthonormal treatment contrasts when there is one outlier. Afterwards, a lot of work has been done in this area by a number of authors (Mandal and Shah 1993; Biswas 2012; Biswas, Das, and Mandal 2013). However, all these works are related to the presence of a single outlier. But, in practice, there may be more than one outlier in many situations affecting the inference process. Bhar, Gupta, and Parsad (2013) considered detection of more than one outlier in designed experiments together with examples. They also worked extensively on this matter and undertook a project entitled "Outliers in Designed Experiments" in 2008 at Indian Agricultural Statistics Research Institute, New Delhi, India (Bhar, Gupta, and Parsad 2008). In this paper, we are concerned with finding a robust design for the estimation of a full set of orthonormal treatment contrasts in the block design set-ups, against presence of more than one outlier. We have observed that in the class of connected, proper variance balanced designs, a robust design does not exist for the estimation of a full set of orthonormal treatment contrasts

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in presence of more than one outlier. We have also observed numerically that as we pass on gradually from balanced complete design to incomplete balanced design and then to unbalanced incomplete design, more and more we deviate from robustness.

2. Block designs robust against presence of more than one outlier

Let us consider the following block design set-up with v treatments in b blocks of sizes $k_1, k_2, \ldots, k_b$:

$$\mathbf{y} = \mu\mathbf{1}_n + \mathbf{D}_1\boldsymbol{\tau} + \mathbf{D}_2\boldsymbol{\beta} + \mathbf{e} \tag{2.1}$$

where $\mathbf{y}(n \times 1)$ is the vector of observations, μ is the mean response, $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ are the vector of treatment effects and block effects respectively; $\mathbf{D}_1(n \times v)$ and $\mathbf{D}_2(n \times b)$ are the observations versus treatments and observations versus blocks incidence matrices respectively. $\mathbf{1}_n$ is the n-component vector with all elements unity and $\mathbf{e}$ is the random error vector with mean vector $\mathbf{0}$ and dispersion matrix $\sigma^2\mathbf{I}_n$ where $\mathbf{I}_n$ is the identity matrix of order n. Let $\mathbf{L}$ be an orthonormal matrix of order v of the form

$$\mathbf{L} = \left(\tfrac{1}{\sqrt{v}}\mathbf{1}_v, \ \mathbf{P}'\right)'$$

Then $\mathbf{P}^{v-1\times v}$ satisfies

$$\mathbf{P1}_v = \mathbf{0}, \mathbf{PP}' = \mathbf{I}_{v-1}, \ \ \mathbf{P}'\mathbf{P} = \mathbf{I}_v - (1/v)\mathbf{1}_v\mathbf{1}'_v \tag{2.2}$$

Suppose we are interested in inferring on a full set of orthonormal treatment contrasts $\boldsymbol{\phi}$ where

$$\boldsymbol{\phi} = \mathbf{P}\boldsymbol{\tau} \tag{2.3}$$

Given a design, the best linear unbiased estimator (BLUE) of $\boldsymbol{\phi}$ is given by $\hat{\boldsymbol{\phi}} = \mathbf{P}\hat{\boldsymbol{\tau}}$ where $\hat{\boldsymbol{\tau}}$ satisfies $\mathbf{C}\hat{\boldsymbol{\tau}} = \mathbf{Q}$. $\mathbf{C}$ and $\mathbf{Q}$ are respectively the information matrix of treatment effects and the vector of adjusted treatment totals where

$$\mathbf{C} = \mathbf{r}^{\delta} - \mathbf{Nk}^{-\delta}\mathbf{N}', \ \ \mathbf{Q} = \mathbf{Gy} \tag{2.4}$$

$$\mathbf{G} = \mathbf{D}_1' - \mathbf{Nk}^{-\delta}\mathbf{D}_2', \ \ \mathbf{N} = \mathbf{D}_1'\mathbf{D}_2, \ \ \mathbf{r}^{\delta} = \mathrm{diag}(r_1, r_2, \ldots, r_v), \ \ \mathbf{k}^{-\delta} = \mathrm{diag}\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b}\right) \tag{2.5}$$

r_i is the replication of the ith treatment and k_j is the size of the jth block. Let us restrict to the class $\mathscr{D}$ of connected block designs so that all treatment contrasts are estimable and rank($\mathbf{C}$)$= v-1$. As a consequence, we have, rank($\mathbf{PCP}'$) $=v-1$. Now using Equations (2.2)–(2.5), we can write the reduced normal equation for the treatment effects $\mathbf{C}\hat{\boldsymbol{\tau}} = \mathbf{Q}$ in the form $(\mathbf{PCP}')(\mathbf{P}\hat{\boldsymbol{\tau}}) = \mathbf{PGy}$ so that

$$\hat{\boldsymbol{\phi}} = \mathbf{P}\hat{\boldsymbol{\tau}} = (\mathbf{PCP}')^{-1}\mathbf{PGy} = \mathbf{Hy}, \ \ \mathbf{H} = (\mathbf{PCP}')^{-1}\mathbf{PG} \tag{2.6}$$

Let there be m $(< n)$ outliers out of n observations $y_1, y_2, \ldots, y_n$, and without loss of generality, let the outliers happen to be with the first m observations. Suppose, the uth observation has added to it an aberration a_u making the uth observation y_u an outlier $u = 1, 2, \ldots, m$. Then the discrepancy $\delta_{i(1,2,\ldots,m)}$ in $\hat{\phi}_i$ due to these m outliers is given by

$$\delta_{i(1,2,\ldots,m)} = h_{i1}a_1 + h_{i2}a_2 + \cdots + h_{im}a_m \tag{2.7}$$

where h_{ij} is the (i,j)th element of $\mathbf{H}$ given by Equation (2.6). Since nothing is known about the relative magnitudes of the m outliers attached to the m observations, it is reasonable to assume that all the a_i s are same i.e., $a_1 = a_2 = \cdots = a_m = a$, say. Then $\delta_{i(1,2,\ldots,m)}$ given by Equation (2.7) simplifies to

$$\delta_{i(1,2,\ldots,m)} = a(h_{i1} + h_{i2} + \cdots + h_{im}) \tag{2.8}$$

Write

$$\boldsymbol{\delta}_{(1,2,\ldots,m)} = \left(\delta_{1(1,2,\ldots,m)}, \delta_{2(1,2,\ldots,m)}, \ldots, \delta_{v-1(1,2,\ldots,m)}\right)'$$

Then following Mandal (1989), the overall discrepancy due to the outliers at the positions $1, 2, \ldots, m$ can be defined as

$$d_{(1,2,\ldots,m)} = \boldsymbol{\delta}'_{(1,2,\ldots,m)}\mathbf{V}\boldsymbol{\delta}_{(1,2,\ldots,m)} = \boldsymbol{\delta}'_{(1,2,\ldots,m)}\mathbf{PCP}'\boldsymbol{\delta}_{(1,2,\ldots,m)} \tag{2.9}$$

where $\text{Disp}(\hat{\phi}) = \sigma^2\mathbf{V}^{-1} = \sigma^2(\mathbf{PCP}')^{-1}$. But the outliers can occur with any m observations $(u_1, u_2, \ldots, u_m)$ among the n observations giving different values of $d_{(u_1,u_2,\ldots,u_m)}$, $1 \le u_1 < u_2 < \cdots < u_m \le n$. The average discrepancy is given by

$$\bar{d} = \frac{1}{\binom{n}{m}} \sum_{1 \le u_1 < u_2 < \cdots < u_m \le n} d_{(u_1,u_2,\ldots,u_m)} \tag{2.10}$$

Following Box and Draper (1975) a convenient measure of uniformity can be taken as

$$s_m = \sqrt{\frac{1}{\binom{n}{m}} \sum_{1 \le u_1 < u_2 < \cdots < u_m \le n} \left(d_{(u_1,u_2,\ldots,u_m)} - \bar{d}\right)^2} \tag{2.11}$$

A design will be called robust in $\mathscr{D}$ when s_m vanishes. We shall see in Subsection 3.1 that $\bar{d}$ is the same for all the designs in $\mathscr{D}$. Then equivalently, a design is robust if all the $d_{(u_1,u_2,\ldots,u_m)}$, $1 \le u_1 < u_2 < \cdots < u_m \le n$, are equal (cf. Box and Draper 1975; Mandal 1989).

In the following section, we will characterize a robust design defined above.

3. Characterization of a robust design

We have defined that a block design is robust in $\mathscr{D}$ if the $d_{(u_1,u_2,\ldots,u_m)}$ s, $1 \le u_1 < u_2 < \cdots < u_m \le n$ are all equal. We first consider in Subsection 3.1, the case of two outliers and then in Subsection 3.2, the general case of m $(< n)$ outliers.

3.1. Robust design with 2 outliers

Without loss of generality, let the outlier a exist with the first two observations y_1 and y_2. Then, using Equation (2.8), the discrepancy in $\hat{\phi}_i$ due to the outliers for the first two observations is given by

$$\delta_{i(12)} = a(h_{i1} + h_{i2}).$$

Writing $\boldsymbol{\delta}_{12} = (\delta_{1(12)}, \delta_{2(12)}, ..., \delta_{(v-1)(12)})'$, the overall discrepancy d_{12} due to the outliers at positions 1 and 2 is given by

$$d_{12} = \boldsymbol{\delta}'_{12}\mathbf{V}\boldsymbol{\delta}_{12} = a^2(\mathbf{h}_1 + \mathbf{h}_2)'\mathbf{V}(\mathbf{h}_1 + \mathbf{h}_2)$$

where $\mathbf{h}_s = (h_{1s}, h_{2s}, ..., h_{v-1,s})'$; $s = 1, 2$. Hence, in general, the overall discrepancy due to the outliers at positions u_1 and u_2 $(1 \leq u_1 < u_2 \leq n)$ is given by

$$d_{u_1u_2} = \boldsymbol{\delta}_{u_1u_2}{}'\mathbf{V}\boldsymbol{\delta}_{u_1u_2} = a^2(\mathbf{h}_{u_1} + \mathbf{h}_{u_2})'\mathbf{V}(\mathbf{h}_{u_1} + \mathbf{h}_{u_2}) \tag{3.1}$$

where $\boldsymbol{\delta}_{u_1u_2} = (\delta_{1(u_1u_2)}, \delta_{2(u_1u_2)}, ..., \delta_{(v-1)(u_1u_2)})'$

Considering $d_{u_1u_2}$ s for all the $\binom{n}{2}$ combinations of $(u_1,\ u_2)$, the total discrepancy of all the $d_{u_1u_2}$ s is given by

$$\begin{aligned}\sum_{u_1<u_2} d_{u_1u_2} = \sum_{u_1<u_2} \boldsymbol{\delta}_{u_1u_2}{}'\mathbf{V}\boldsymbol{\delta}_{u_1u_2} &= a^2 \sum_{u_1<u_2} (\mathbf{h}_{u_1} + \mathbf{h}_{u_2})'\mathbf{V}(\mathbf{h}_{u_1} + \mathbf{h}_{u_2}) \\ &= a^2\left[(n-1)\sum_{i=1}^{n}(\mathbf{h}_i{}'\mathbf{V}\mathbf{h}_i) + 2\sum_{1\leq i<j\leq n}(\mathbf{h}_i{}'\mathbf{V}\mathbf{h}_j)\right]\end{aligned} \tag{3.2}$$

It is to be noted that

$$\left(\sum_{i=1}^{n}\mathbf{h}_i\right)'\mathbf{V}\left(\sum_{j=1}^{n}\mathbf{h}_j\right) = \sum_{i=1}^{n}\mathbf{h}_i{}'\mathbf{V}\mathbf{h}_i + 2\sum_{1\leq i<j\leq n}\mathbf{h}_i{}'\mathbf{V}\mathbf{h}_j \tag{3.3}$$

From Equations (2.5) and (2.6) it follows that

$$\mathbf{H1} = \sum_{i=1}^{n}\mathbf{h}_i = \mathbf{0} \tag{3.4}$$

From Equation (3.3) and (3.4) we get

$$tr.(\mathbf{H}'\mathbf{V}\mathbf{H}) = \sum_{i=1}^{n}\mathbf{h}_i'\mathbf{V}\mathbf{h}_i = -2\sum_{1\leq i<j\leq n}\mathbf{h}_i'\mathbf{V}\mathbf{h}_j \tag{3.5}$$

Hence, from Equations (3.2), (3.4), and (3.5) we have

$$\sum_{u_1<u_2} d_{u_1u_2} = (n-2)a^2 tr.(\mathbf{H}'\mathbf{V}\mathbf{H}) \tag{3.6}$$

Using the expressions of $\mathbf{V}$ and $\mathbf{H}$ and the relation $\mathbf{GG}' = \mathbf{C}$, we derive that

$$tr.(\mathbf{H}'\mathbf{V}\mathbf{H}) = v - 1 \tag{3.7}$$

From Equations (3.6) and (3.7) it follows that the total discrepancy is independent of the designs in $\mathscr{D}$. So s_m defined in Equation (2.11) is not affected by the average discrepancy of the designs in $\mathscr{D}$. So, as in the case of a single outlier, robust design is the one for which $d_{u_1u_2}$ s are all equal.

3.2. Robust design with m (<n) outliers

We now consider the general case of $m(< n)$ outliers. If the outliers are associated with first m observations, then the overall discrepancy $d_{12\ldots m}$ is given by

$$a^2(\mathbf{h}_1 + \mathbf{h}_2 + \cdots + \mathbf{h}_m)'\mathbf{V}(\mathbf{h}_1 + \mathbf{h}_2 + \cdots + \mathbf{h}_m)$$

And if the outliers occur at $(u_1, u_2, \ldots, u_m)th$ position, then the overall discrepancy is given by

$$\begin{aligned} d_{u_1u_2\ldots u_m} &= \boldsymbol{\delta}'_{u_1,u_2,\ldots,u_m}\mathbf{V}\boldsymbol{\delta}_{u_1,u_2,\ldots,u_m} \\ &= a^2(\mathbf{h}_{u_1} + \mathbf{h}_{u_2} + \cdots + \mathbf{h}_{u_m})'\mathbf{V}(\mathbf{h}_{u_1} + \mathbf{h}_{u_2} + \cdots + \mathbf{h}_{u_m}); \\ & 1 \le u_1 < u_2 < \cdots < u_m \le n \end{aligned} \tag{3.8}$$

$$\begin{aligned} \text{Total discrepancy} &= \sum_{1\le u_1<u_2<\cdots<u_m\le n} d_{u_1u_2\ldots u_m} \\ &= a^2\left[\binom{n-1}{m-1}\sum_{i=1}^{n}\mathbf{h}'_i\mathbf{V}\mathbf{h}_i + 2\binom{n-2}{m-2}\sum_{1\le i<j\le n}\mathbf{h}'_i\mathbf{V}\mathbf{h}_j\right] \end{aligned} \tag{3.9}$$

Now from Equations (3.2) and (3.4) and (3.5), the right-hand side of Equation (3.9) can be reduced to

$$\begin{aligned} &= a^2\left[\binom{n-2}{m-2}\left(\sum_{i=1}^{n}\mathbf{h}_i\right)'\mathbf{V}\left(\sum_{i=1}^{n}\mathbf{h}_i\right) + \left(\binom{n-1}{m-1} - \binom{n-2}{m-2}\right)\sum_{i=1}^{n}(\mathbf{h}'_i\mathbf{V}\mathbf{h}_i)\right] \\ &= a^2\left[\binom{n-2}{m-2}\mathbf{1}'\mathbf{H}'\mathbf{V}\mathbf{H}\mathbf{1} + \binom{n-2}{m-1}tr.(\mathbf{H}'\mathbf{V}\mathbf{H})\right] \\ &= a^2\binom{n-2}{m-1}tr.(\mathbf{H}'\mathbf{V}\mathbf{H}) \ \ (\text{since } \mathbf{H1} = \mathbf{0}) \\ &= a^2(v-1)\binom{n-2}{m-1} \end{aligned}$$

In the general case of m $(< n)$ outliers, also we see that the average discrepancy is independent of the designs in $\mathscr{D}$. So a design is robust in $\mathscr{D}$ if $d_{u_1u_2\ldots u_m}$ given in Equation (3.8); $1 \le u_1 < u_2 < \cdots < u_m \le n$ s are all equal for $\binom{n}{m}$ choices of $u_1, u_2, \ldots, u_m$. Writing all the equations in terms of $\mathbf{h}$'s elaborately and using Equation (3.5), it follows that robustness can be realized if $\mathbf{H}'\mathbf{V}\mathbf{H}$ is completely symmetric (c.s.) i.e., the diagonal elements are all equal and off-diagonal elements of $\mathbf{H}'\mathbf{V}\mathbf{H}$ are all equal. Thus we get the following theorem.

Theorem 3.1. *In the class of all connected block designs $\mathscr{D}$, if there exists a design for which $\mathbf{H}'\mathbf{V}\mathbf{H}$ is c.s. where $\mathbf{H}$ is given by Equation (2.6), then it is robust for the estimation of a full set of orthonormal treatment contrasts.*

From the expression of $\boldsymbol{\delta}'_{u_1,u_2,\ldots,u_m}\mathbf{V}\boldsymbol{\delta}_{u_1,u_2,\ldots,u_m}$ in Equation (3.8), it is clear that for $m = 1$, it simply reduces to $\boldsymbol{\delta}'_u\mathbf{V}\boldsymbol{\delta}_u$ if the outlier is present in the u^{th} observation; $1 \le u \le n$. And the condition of equality of the $\boldsymbol{\delta}'_u\mathbf{V}\boldsymbol{\delta}_u$ s for robustness, it reduces to the equality of the diagonal elements of $\mathbf{H}'\mathbf{V}\mathbf{H}$ only.

4. Nonexistence of robust design

In Section 3, we have observed that in the class of all connected block designs, if there exists a design for which $\mathbf{H}'\mathbf{V}\mathbf{H}$ is c.s., then it is robust for the estimation of a full set of orthonormal treatment contrasts.

Without loss of generality, we can take

$$\mathbf{P} = [\xi_1, \xi_2, ..., \xi_{v-1}]' \tag{4.1}$$

where $\xi_1, \xi_2, ..., \xi_{v-1}$ are orthonormal eigenvectors corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, ..., \lambda_{v-1}$ of the $\mathbf{C}$ matrix respectively. Then

$$\mathbf{PCP}' = \Lambda = \text{diag}[\lambda_1, \lambda_2, ..., \lambda_{v-1}], \;\; \mathbf{H} = \Lambda^{-1}\mathbf{PG} \tag{4.2}$$

Now

$$\mathbf{H}'\mathbf{VH} = \mathbf{G}'\mathbf{P}'\Lambda^{-1}\mathbf{PG}. \tag{4.3}$$

However it is difficult to find a robust design in $\mathscr{D}$ for the general set up. So we restrict our search to the subclass $\mathscr{D}_0$ of connected, proper variance balanced designs (VBDs) to find a robust design. Then for connected, proper VBD, $k_1 = k_2 = \cdots = k_b = k$, say, and $\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1} = \lambda$, say. Hence from Equations (2.5) and (4.3), we get

$$\begin{aligned} \mathbf{H}'\mathbf{VH} &= \mathbf{G}'\mathbf{P}'\Lambda^{-1}\mathbf{PG} \\ &= \lambda^{-1}\mathbf{G}'\mathbf{G} \quad (\text{Since } \mathbf{G1} = \mathbf{0}) \\ &= \lambda^{-1}\left(\mathbf{D}_1\mathbf{D}'_1 - \frac{\mathbf{D}_1\mathbf{N}\mathbf{D}'_2}{k} - \frac{\mathbf{D}_2\mathbf{N}'\mathbf{D}'_1}{k} + \frac{\mathbf{D}_2\mathbf{N}'\mathbf{N}\mathbf{D}'_2}{k^2}\right) \end{aligned}$$

Note that in each row of $\mathbf{D}_1$ and $\mathbf{D}_2$, only one element is non-zero, the rest are zero. Then the (u, u')th element of $\mathbf{D}_1\mathbf{N}\mathbf{D}'_2$ is n_{ij} if u corresponds to ith treatment and u' corresponds to j^{th} block and the corresponding element in $\mathbf{D}_2\mathbf{N}'\mathbf{N}\mathbf{D}'_2$ is $\mu_{jj'}$, where $\mu_{jj'}$ is the number of treatments common between block j (corresponding to unit u) and j' (corresponding to unit u'). (u, u')th element of $\mathbf{D}_1\mathbf{D}'_1$ is 1 if u and u'^{th} observation is obtained by using same treatment and is zero otherwise. Hence the off-diagonal elements of $\mathbf{D}_1\mathbf{D}'_1$ take values both 0 and 1 depending on the observations. Hence $\mathbf{H}'\mathbf{VH}$ is not c.s. in the class of connected, proper variance balanced designs. Thus a connected, proper variance balanced design is not robust for the estimation of a full set of orthonormal treatment contrasts if there exist more than one outlier. Hence we get following theorem.

Theorem 4.1. *In the class $\mathscr{D}_0$ of connected, proper variance balanced designs, a robust design does not exist for the estimation of a full set of orthonormal treatment contrasts in presence of more than one outlier.*

Now we consider the following example where we observe that as we deviate from regular designs, the measure of deviation from uniformity s_m of $d_{u_1, u_2, ..., u_m}$'s introduced in Section 2 increases.

5. Example

Though we observe that even in the restricted subclass mentioned in Theorem 4.1, a robust design does not exist, we below study numerically with two aberrations, how the values of the criterion defined in Equation (2.11) measuring the departure from robustness vary over different designs with same v and r.

We consider a randomized block design (RBD) with $v = 7$ and $r = 4$; a symmetric balance incomplete block design (SBIBD) with v=7, r=4, λ=2; and a binary, Proper, Equireplicate, Incomplete Block design (BPEIBD) with b= v=7, r=4. The blocks of SBIBD(7,4,2) are:

Block1: (1,4,6,7); Block2: (2,5,7,1); Block3: (3,6,1,2); Block4: (4,7,2,3); Block5: (5,1,3,4); Block6: (6,2,4,5); Block7: (7,3,5,6).

Now we construct BPEIBD (4, 7) from the above SBIBD(7,4,2) by interchanging treatment 5 of Block 2 with treatment 3 of Block 3 and other blocks are kept fixed.

We use the measure of deviation from uniformity of $d_{(u_1, u_2, \ldots, u_m)}$ s given in Equation (2.11). In this example we consider two outliers with equal aberration a=1.

Hence measure of deviation from uniformity is given by $s_2 = \sqrt{\frac{2}{n(n-1)} \sum_{1 \le u_1 < u_2 \le n} (d_{(u_1, u_2)} - \bar{d})^2}$. Using Equation (2.5), **C** matrices of (A1), (A2), (A3) in the Appendix, Equations (2.9) and (2.10) we construct the following table for s_2 :

	RBD	BIBD	BPEIBD
s_2	0.1191	0.1700	0.1790

We observe from the above table that as we move from RBD to SBIBD, and further to BPEIBD, the value of s_2 increases. Though we have given this table for one set of v and r, but from further computations, we have observed that this is true for other combinations of (v, r) as well. Thus we can conjecture that the Theorem 4.1 holds true even outside the ambit of the class of proper variance balanced design.

6. Concluding Remarks

In this paper, in the block design set up, we have characterized designs which are robust against presence of more than one outlier in the observations for the estimation of a full set of orthonormal treatment contrasts. It is known that a Balanced incomplete block design, a Partially balanced incomplete block design with certain properties are robust for the estimation of a full set of orthonormal treatment contrasts in the presence of a single outlier (cf. Mandal 1989; Mandal and Shah 1993). But if the number of outliers is more than one, even a Randomized block design or a Balanced incomplete block design is not robust for the same estimation problem. This is because the requirement of complete symmetry property of $\mathbf{H'VH}$ is not attainable. But for the single outlier case, we need only the equality of the diagonal elements of $\mathbf{H'VH}$ and this can be realized by some standard designs. Though no robust design exists for the case of multiple outliers in the sense of uniformity of $d_{u_1, u_2, \ldots, u_m}$ values, but numerical computations show that for multiple outliers, more we deviate from regularity in designs, the more we deviate from robustness. The finding is important since in practice, the number of outliers may well be more than one in any practical situation.

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References

Bhar, L., V. K. Gupta, and R. Parsad. 2013. Detection of outliers in designed experiments in presence of masking. *Statistics and Applications* 11:147–60.

Bhar, L., V. K. Gupta, and R. Parsad. 2008. *Outliers in Designed Experiments*. New Delhi, India: Indian Agricultural Statistics Research Institute.

Biswas, A. 2012. Block designs robust against the presence of an aberration in a treatment control set up. *Communications in Statistics – Theory and Methods* 41 (5):920–33. doi:10.1080/03610926.2010.530372.

Biswas, A., P. Das, and N. K. Mandal. 2013. Designs robust against the presence of an aberration in a diallel cross design set up. *Communications in Statistics – Theory and Methods* 42 (15): 2810–17. doi:10.1080/03610926.2011.620210.

Box, G. E. P., and N. R. Draper. 1975. Robust designs. *Biometrika* 62 (2):347–52. doi:10.1093/biomet/62.2.347.

Mandal, N. K. 1989. On robust designs. *Calcutta Statistical Association Bulletin* 38 (1–2):115–19. doi:10.1177/0008068319890109.

Mandal, N. K., and K. R. Shah. 1993. Designs robust against aberrations. *Calcutta Statistical Association Bulletin* 43 (1–2):95–107. doi:10.1177/0008068319930109.

Appendix

For RBD(7,4), SBIBD(7,4,2) and BPEIBD (7,7,4), **C** matrices are respectively given by:

$$\mathbf{C}_{RBD} = \begin{pmatrix} 3.4286 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 \\ -0.5714 & 3.4286 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 \\ -0.5714 & -0.5714 & 3.4286 & -0.5714 & -0.5714 & -0.5714 & -0.5714 \\ -0.5714 & -0.5714 & -0.5714 & 3.4286 & -0.5714 & -0.5714 & -0.5714 \\ -0.5714 & -0.5714 & -0.5714 & -0.5714 & 3.4286 & -0.5714 & -0.5714 \\ -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & 3.4286 & -0.5714 \\ -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & -0.5714 & 3.4286 \end{pmatrix} \quad \text{(A1)}$$

$$\mathbf{C}_{SBIBD} = \begin{pmatrix} 3.0 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & 3.0 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 3.0 & -0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 3.0 & -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 & 3.0 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 3.0 & -0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 3.0 \end{pmatrix} \quad \text{(A2)}$$

$$\mathbf{C}_{BPEIBD} = \begin{pmatrix} 3.0 & -0.5 & -0.50 & -0.5 & -0.50 & -0.50 & -0.50 \\ -0.5 & 3.0 & -0.50 & -0.5 & -0.50 & -0.50 & -0.50 \\ -0.5 & -0.5 & 3.00 & -0.5 & -0.50 & -0.25 & -0.75 \\ -0.5 & -0.5 & -0.50 & 3.0 & -0.50 & -0.50 & -0.50 \\ -0.5 & -0.5 & -0.50 & -0.5 & 3.00 & -0.75 & -0.25 \\ -0.5 & -0.5 & -0.25 & -0.5 & -0.75 & 3.00 & -0.50 \\ -0.5 & -0.5 & -0.75 & -0.5 & -0.25 & -0.50 & 3.00 \end{pmatrix} \quad \text{(A3)}$$